

# The Milnor fiber of the singularity

$$f(x, y) + zg(x, y) = 0$$

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## Abstract

We give a description of the Milnor fiber and the monodromy of a singularity of the form  $f + zg = 0$  where  $f$  and  $g$  define plane curves and have no common components. The description depends only on the topological type of the two plane curve germs defined by  $f$  and  $g$ . In particular, this gives a description of the boundary of the Milnor fiber. As a corollary, we give a simple formula for the monodromy zeta function and the Euler characteristic of the fiber.

**Keywords.** nonisolated hypersurface singularities, Milnor fiber, monodromy zeta function

## 1 Introduction

Let  $\Phi : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$ ,  $(x, y, z) \mapsto f(x, y) + zg(x, y)$  be a function germ, where  $f, g : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ . We will only require that  $f$  and  $g$  have no common factors. This singularity is not isolated; the singular set is the  $z$ -axis. We determine the diffeomorphism type of the Milnor fiber  $F_\Phi$  in terms of a simultaneous embedded resolution graph of  $f$  and  $g$ , and some information about the monodromy. In particular, we get a description of the boundary  $\partial F_\Phi$ .

Singularities of this type play an important role in the investigations of sandwiched singularities, as described in [1].

In section 2 we recall some topological properties of hypersurface singularities with emphasis on non-isolated singularities and the singular Milnor fibre. We then recall some properties of plane curve singularities. Finally, we recall the notion of a 4 dimensional handlebody and fix some notation for surgery.

In 3 we construct a subset  $T_{f,g}$  of a common embedded resolution of  $f$  and  $g$  from tubular neighbourhoods around some divisors. We obtain the space  $F_{f,g}$  by performing surgery along certain embedded disks in  $T_{f,g}$ . This surgery

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does not change the homotopy type. Our main theorem states that  $F_{f,g}$  has the same diffeomorphism type as Milnor fibre  $F_\Phi$ . Furthermore,  $F_{f,g}$  can be decomposed into a union of sets on which the monodromy can be described completely. As a corollary, we obtain a simple formula for the monodromy zeta function and the Euler characteristic  $\chi(F_\Phi)$ .

Section 4 contains a proof of the main statement 3.3 from section 3.

## 2 Hypersurface singularities

### 2.1 General results

In this subsection we recall some of the general properties of the Milnor fiber of a holomorphic germ  $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ , the monodromy associated to such a germ, and other invariants related to these two.

Let  $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$  be a hypersurface singularity, denote by  $B_\delta$  the closed ball with radius  $\delta$  around the origin in  $\mathbb{C}^{n+1}$ , and by  $D_\epsilon$  the closed disk around the origin in  $\mathbb{C}$  with radius  $\epsilon$ .  $D$  will denote an arbitrary closed disk in the complex plane. We let  $V_f = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$  and  $S_f = \{z \in \mathbb{C}^{n+1} : \partial f = 0\}$ . The link of  $f$  is defined as  $K = V_f \cap \partial B_\delta$  for  $0 < \delta \ll 1$ .

The Milnor fiber  $F_f$  of  $f$  is by definition the fiber  $f^{-1}(\epsilon) \cap B_\delta$  for  $0 < \epsilon \ll \delta \ll 1$ . Then  $F_f$  is a smooth  $2n$  dimensional manifold, and so has the homotopy type of a CW complex. In [4], Milnor proves that  $F_f$  is homotopy equivalent to a finite  $n$ -dimensional CW-complex. Moreover, if  $s$  is the dimension of the singular locus  $S_f$ , then  $F_f$  is  $(n - s - 1)$ -connected, as proved in [3].

Let  $E = f^{-1}(\partial D_\epsilon) \cap B_\delta$ . The function  $E \rightarrow \partial D_\epsilon$ ,  $z \mapsto f(z)$  is a locally trivial fiber bundle with fiber  $F_f$ . If  $T = \{z \in \partial B_\delta : |f(z)| < \epsilon\}$ , we can define another fiber bundle  $\partial B_\delta \setminus T \rightarrow \partial D_1$ ,  $z \mapsto f(z)/|f(z)|$ . These two fiber bundles are isomorphic. In fact, there is a bundle-isomorphism  $E \rightarrow \partial B \setminus T$  which restricts to the identity on  $\partial T$ . In particular, we have a diffeomorphism

$$F_f \cong \{z \in \partial B_\delta \setminus T : f(z)/|f(z)| = 1\}. \quad (1)$$

The singular fiber of  $f$  is defined as

$$F_{f,sing} = \{z : |z| = \delta, |f(z)| > 0, f(z)/|f(z)| = 1\} \cup K.$$

Usually,  $F_{f,sing}$  is not a smooth manifold. By the description 1 of  $F_f$  we have an inclusion  $\iota : F_f \hookrightarrow F_{f,sing}$ . If  $f$  is an isolated singularity,  $\iota$  is a homotopy equivalence, as proved in [4]. For non-isolated singularities this does generally not hold.

### 2.2 The zeta function of the monodromy

The monodromy of the Milnor fibration is a diffeomorphism  $m_f : F \rightarrow F$  with the property that this bundle is isomorphic to the bundle given by  $F \times I / ((p, 0) \sim (m_f(p), 1)) \rightarrow I / (0 \sim 1)$ ,  $(p, t) \mapsto t$ . The monodromy is determined by the bundle up to isotopy, and the bundle is determined up to bundle isomorphism by the monodromy. The monodromy induces linear isomorphisms  $h_i : H_i(F; \mathbb{C}) \rightarrow H_i(F; \mathbb{C})$ .

We call the product

$$\zeta_f(t) = \prod_{i=0}^{\infty} \det(I - th_i)^{(-1)^{i+1}}$$

the zeta function associated with the singularity  $f$ . This product is well defined because  $F$  is a finite CW complex, and so  $\dim_{\mathbb{C}} H_*(F; \mathbb{C}) < \infty$ . The zeta function behaves multiplicatively in the following sense:

Let  $C$  be a subset of  $F$  so that  $\dim H_*(C; \mathbb{C}) < \infty$  and  $m_f$  restricts to a homeomorphism  $m_C : C \rightarrow C$ . Let us call such a subset good with respect to  $m$ . Then  $m_C$  induces a linear automorphism  $h_{C,i}$  on  $H_i(C; \mathbb{C})$  and we define

$$\zeta_C(t) = \prod_{i=0}^{\infty} \det(I - th_{C,i})^{(-1)^{i+1}}.$$

The following propositions are well known:

**Proposition 2.1.** *Assume that  $A, B \subset F$  so that  $A, B, A \cap B$  are good subsets of  $F$  and the interiors of  $A$  and  $B$  cover  $F$ . Then we have  $\zeta_f(t) = \zeta_A(t)\zeta_B(t)\zeta_{A \cap B}(t)^{-1}$ .  $\square$*

**Proposition 2.2.** *We have  $\chi(F) = \nu(\zeta_f)$ , where  $\nu : \mathbb{C}(t)^{\times} \rightarrow \mathbb{Z}$  is the valuation at infinity.  $\square$*

The monodromy  $m_f$  can be extended to a homeomorphism  $m_{f,sing} : F_{f,sing} \rightarrow F_{f,sing}$ , which is called the singular monodromy. In fact, by defining  $F_{f,sing,\theta}$  in the same way as  $F_{f,sing}$ , only replacing the condition  $f/|f| = 1$  by  $f/|f| = \theta$ , we get a subspace  $\cup_{\theta} F_{f,sing,\theta} \times \{\theta\} \subset S^{2n+1} \times S^1$ . The projection onto  $S^1$  is a locally trivial fiberbundle with fiber  $F_{f,sing}$ ; its monodromy is the singular monodromy.

## 2.3 Plane curves

In the case  $n = 1$ ,  $f$  is a plane curve singularity. For a detailed introduction, see [5]. Write  $f = f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_k^{\alpha_k}$  where  $f_1, \dots, f_k$  are the  $k$  different irreducible factors of  $f$ . In this case,  $K$  is a link in  $\partial B_{\delta}$ . Let  $T$  be a tubular neighbourhood around  $K$  and  $\bar{T}$  the corresponding closed tubular neighbourhood. There exists a projection  $c : \bar{T} \rightarrow K$  which is a trivial  $D$ -bundle, this is just the normal bundle of the link. Further, write  $K = \cup_{i=1}^k K_i$ , where  $K_i = \{z \in \partial B_{\delta} : f_i = 0\}$ , and  $T = \cup_{i=1}^k T_i$ , where  $T_i$  is the component of  $T$  containing  $K_i$ . Choosing  $\epsilon > 0$  small enough, we can choose  $T = \{z \in \partial B_{\delta} : |f(z)| < \epsilon\}$ . Then  $\partial F_f \subset \partial \bar{T}$ . The projection  $c$  can be chosen in such a way that the restriction  $c_i = c|_{F_f \cap \partial \bar{T}_i} : F_f \cap \partial \bar{T}_i \rightarrow K_i$  is a covering map. This map can be described in terms of the embedded resolution graph of  $f$ ; we recall some of its properties.

Let  $\Gamma_f = (\mathcal{V}, \mathcal{E})$  be the embedded resolution graph of some fixed embedded resolution of  $f$  (see [5] for definition and properties). Here  $\mathcal{V}$  is the set of vertices and  $\mathcal{E}$  the set of edges. Write  $\mathcal{V} = \mathcal{W} \amalg \mathcal{A}_f$  where  $\mathcal{A}_f$  consists of the arrowhead vertices of  $\Gamma$  and  $\mathcal{W}$  consists of the nonarrowhead vertices. The elements of  $\mathcal{A}_f$  correspond to the branches of  $f$  so there is a natural correspondence between the arrowhead vertices of  $\Gamma$  and the components of

$K$ . We will make no distinction between the indices  $i = 1, \dots, k$  and the corresponding  $a \in \mathcal{A}_f$ .

For each  $a \in \mathcal{A}_f$  there exists a unique  $w_a \in \mathcal{W}$  so that  $(w_a, a) \in \mathcal{E}$ . The map  $f$  has multiplicity  $\alpha_a$  on  $a$ , let  $m_{w_a}$  be its multiplicity on  $w_a$ . Then  $F_f \cap T_a$  has  $(\alpha_a, m_{w_a})$  components, and restricting  $c_a$  to any of these components gives a covering of degree  $\alpha_a/(\alpha_a, m_{w_a})$ . The singular fiber  $F_{f, \text{sing}}$  of  $f$  is homeomorphic to the space  $F_f/\sim$  where the equivalence relation  $\sim$  is given by  $z_1 \sim z_2$  if and only if  $z_1, z_2 \in F_f \cap T_i$  for some  $a$ , and  $c_a(z_1) = c_a(z_2)$ .

The monodromy  $m_f : F_f \rightarrow F_f$  can be chosen so that it preserves this equivalence relations, that is,  $x_1 \sim x_2$  if and only if  $m(x_1) \sim m(x_2)$ . Therefore, we get a homeomorphism  $F_{f, \text{sing}} \rightarrow F_{f, \text{sing}}$  induced by the monodromy. This homeomorphism coincides with the singular monodromy already constructed. Note that  $F_{f, \text{sing}} = F_f \cup B$  where both  $B$  and  $F_f \cap B$  are homotopically equivalent to the disjoint union of copies of  $S^1$  (here, the set  $B$  is a disjoint union of sets of the form  $S^1 \times R$  where  $R$  is a union of segments in the plane with one endpoint at the origin). Moreover, these homotopy  $S^1$ 's contract to actual copies of oriented  $S^1$ 's. The singular monodromy  $m_{f, \text{sing}}$  restricts to a homeomorphism  $F_f \rightarrow F_f$  which coincides with the monodromy  $m_f$ . Also,  $m_{f, \text{sing}}$  permutes the connected components of  $B$  and  $F_f \cap B$ , respecting the orientation on the first homology. Thus, the induced maps on the homologies of  $B$  and  $F_f \cap B$  are zero in degree  $> 1$ , and can be represented by the same permutation matrix in degrees zero and one. These cancel out to give  $\zeta_B(t) = \zeta_{F_f \cap B}(t) = 1$ , and therefore, by 2.1,

**Proposition 2.3.** *If  $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$  defines a plane curve singularity, then  $\zeta_f(t) = \zeta_{f, \text{sing}}(t)$ .*

## 2.4 Handles and surgery

We will use handles to describe the Milnor fiber. More precisely, we will use 4 dimensional handles of index 2 in our construction. Chapter 4 of [2] gives a good presentation of the necessary theory.

Let  $X$  be a 4-manifold with boundary and  $\phi : (\partial D) \times D \rightarrow \partial X$  an embedding. We obtain a new manifold  $X \cup_\phi D \times D$  by taking the disjoint union  $X \amalg (D \times D)$  and then identifying any point  $x \in (\partial D) \times D$  with  $\phi(x) \in \partial X$ . The map  $\phi$  induces an isomorphism between the normal bundles of  $(\partial D) \times \{0\}$  in  $(\partial D) \times D$  and  $\phi((\partial D) \times \{0\})$  in  $\partial X$ . Since  $(\partial D) \times \{0\} \subset (\partial D) \times D$  already comes with a canonical framing, this isomorphism can be specified by a framing on  $\phi((\partial D) \times \{0\})$ . The diffeomorphism type of the resulting manifold is determined by the following data (see for example [2]):

- The embedding  $\phi|_{(\partial D) \times \{0\}}$  of  $(\partial D) \times \{0\} \cong S^1$  into  $\partial X$ .
- The framing of the normal bundle of  $\phi|_{(\partial D) \times \{0\}}$ .

We will now fix some notation for surgery along embedded disks. We will assume that all maps respect orientation when appropriate. Let  $X$  be an oriented 4 dimensional manifold with boundary and  $\iota : \bar{D} \hookrightarrow X$  an embedding of the closed disk. We assume that the boundary  $\partial \bar{D}$  is embedded into the

boundary  $\partial X$ , and that  $\iota(\bar{D})$  is transversal to  $\partial X$ . We can find a parametrisation  $\psi : \bar{D} \times \bar{D} \rightarrow X$  of a closed tubular neighbourhood of  $\iota(\bar{D})$  so that  $\psi(0, z) = \iota(z)$ , and  $\psi|_{\bar{D} \times \partial \bar{D}}$  is a parametrisation of a tubular neighbourhood of  $\iota(\partial \bar{D}) \subset \partial X$ . Define  $X' = X \setminus \psi(D \times \bar{D})$ . For  $k \in \mathbb{Z}$  let  $X_{\iota, k} = X' \amalg_{t_k} \bar{D} \times \bar{D}$ , where the glueing map  $t_k : \bar{D} \times \partial \bar{D} \rightarrow X'$  is given by  $t_k(x, y) = \psi(x, x^k y)$ .

**Definition 2.4.** We call  $X_{\iota, k}$  constructed above the  $k$ -th twist of  $X$  along  $\iota(\bar{D})$ .

Note that  $X_{\iota, k}$  is obtained by thinking of  $\psi(\bar{D} \times \bar{D})$  as a handle, removing it, and then attaching it again via a different glueing map. This construction is very similar to Dehn surgery. In fact,  $\partial X_{\iota, k}$  is nothing else than  $\partial X$ , to which a Dehn surgery with coefficient  $1/k$  has been applied along  $\iota(\partial \bar{D})$ .

### 3 Description of the fiber

Let  $f, g : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$  be any plane curve singularities without common factors and define

$$\Phi(x, y, z) = f(x, y) + zg(x, y).$$

Consider a fixed common embedded resolution  $\phi : V \rightarrow \mathbb{C}^2$  of  $f$  and  $g$ . The resolution graph of this embedded resolution will be denoted by  $\Gamma$ . Denote its set of vertices by  $\mathcal{V}(\Gamma)$  and the set of edges by  $\mathcal{E}(\Gamma)$ . We write  $\mathcal{V}(\Gamma) = \mathcal{W}(\Gamma) \amalg \mathcal{A}(\Gamma)$  where  $\mathcal{W}(\Gamma)$  is the set of non-arrowhead vertices and  $\mathcal{A}(\Gamma)$  the set of arrowhead vertices. We decompose  $\mathcal{A}(\Gamma)$  further as  $\mathcal{A}(\Gamma) = \mathcal{A}_f(\Gamma) \amalg \mathcal{A}_g(\Gamma)$ , where the elements of  $\mathcal{A}_f(\Gamma)$  and  $\mathcal{A}_g(\Gamma)$  correspond to components of the strict transform of  $f$  and  $g$  respectively. A vertex  $v \in \mathcal{V}(\Gamma)$  corresponds to a component  $E_v$  of the exceptional divisor  $\phi^{-1}(0)$ , or the strict transform of  $f$  or  $g$ . In each case, we denote by  $m_v$  the multiplicity of  $f$  on  $E_v$ , and  $l_v$  the multiplicity of  $g$  on  $E_v$ . In particular,  $m_v = 0$  if and only if  $v \in \mathcal{A}_g$  and  $l_v = 0$  if and only if  $v \in \mathcal{A}_f$ .

Let  $f' = f \circ \phi$ ,  $g' = g \circ \phi$  and  $F'_f = (f')^{-1}(\epsilon) \cap \phi^{-1}(B_\delta) = \phi^{-1}(F_f)$ . The map  $V \setminus \phi^{-1}(0) \rightarrow \mathbb{C}^2 \setminus \{(0, 0)\}$ ,  $r \rightarrow \phi(r)$  is a diffeomorphism. In particular, it restricts to a diffeomorphism  $F'_f \rightarrow F_f$ .

We have a map  $\phi \times \text{id}_{\mathbb{C}} : V \times \mathbb{C} \rightarrow \mathbb{C}^3$  which restricts to a diffeomorphism  $(V \setminus \phi^{-1}(0)) \times \mathbb{C} \rightarrow \mathbb{C}^3 \setminus \{(0, 0, z) : z \in \mathbb{C}\}$ . We set  $\Phi' = \Phi \circ (\phi \times \text{id}_{\mathbb{C}})$ , and  $F' = (\phi \times \text{id}_{\mathbb{C}})^{-1}(F)$ . Clearly,  $F'$  is diffeomorphic to  $F$ .

For each  $w \in \mathcal{W}$ , choose a small tubular neighbourhood  $T_w$  around  $E_w$  in  $V$  and a map  $b_w : T_w \rightarrow E_w$  which is a smooth open disk bundle. Denote by  $\bar{T}_w$  the corresponding closed tubular neighbourhood. These can be chosen so that they satisfy the following property:

If  $w, w' \in \mathcal{W}$  and  $(w, w') \in \mathcal{E}$ , then we have  $b_w^{-1}(E_w \cap E_{w'}) = E_{w'} \cap T_w$  and  $b_{w'}^{-1}(E_w \cap T_{w'}) = T_w \cap T_{w'}$ . If  $w \in \mathcal{W}$ ,  $a \in \mathcal{A}$  and  $(w, a) \in \mathcal{E}$ , then  $b_w^{-1}(E_w \cap E_a) = E_a \cap T_w$ . Then the set  $T = \cup_{w \in \mathcal{W}} T_w$  is the plumbed 4-manifold with plumbing graph  $\Gamma$ .

If  $w, w' \in \mathcal{W}(\Gamma)$  and  $e = (w, w') \in \mathcal{E}(\Gamma)$ , then we let  $T_e = T_w \cap T_{w'}$ . If  $w \in \mathcal{W}(\Gamma)$  and  $a \in \mathcal{A}(\Gamma)$  so that  $e = (w, a) \in \mathcal{E}(\Gamma)$ , then we pick a small disk-shaped neighbourhood  $U_a$  in  $E_w$  around  $E_w \cap E_a$  and let  $T_a = T_e = b_w^{-1}(U_a)$ . Then  $T_a$  is a tubular neighbourhood around  $E_a$  in  $T$ .

The Milnor fiber  $F_\Phi$  can be described in terms of the embedded resolution graph  $\Gamma$ , with the additional arrowhead vertices, and all vertices decorated by the multiplicities of  $f'$  and  $g'$ . This description will depend on which of the two functions  $f'$  and  $g'$  has higher multiplicities on the exceptional divisors. The following definition makes this precise.

**Definition 3.1.** • Let  $\mathcal{W}_1 = \{w \in \mathcal{W}(\Gamma) : m_w \leq l_w\}$  and  $\mathcal{W}_2 = \mathcal{W} \setminus \mathcal{W}_1$ . Let  $\Gamma_i$  be the subgraph of  $\Gamma$  generated by the set  $\mathcal{W}_i$ . Define  $T_i = \bigcup_{w \in \mathcal{W}_i} T_w$ .

- Let  $\mathcal{A}_{f,i} = \{a \in \mathcal{A}_f : w_a \in \mathcal{W}_i\}$  and  $T_{f,i} = \bigcup_{a \in \mathcal{A}_{f,i}} T_a$ . Repeat this with  $f$  replaced by  $g$ .
- Choose a small  $\epsilon > 0$  and let  $T_\epsilon$  be a small tubular neighbourhood around  $f'^{-1}(\epsilon) \cap \overline{T}$ .
- Let  $T'$  be a small tubular neighbourhood around the exceptional divisor inside  $T$ . This is chosen after choosing  $\epsilon$ . In particular,  $\overline{T'} \cap \overline{T}_\epsilon = \emptyset$ .
- Let  $\overline{T}_{f,g} = [\overline{T}_{f,1} \setminus T'] \cup \overline{T}_\epsilon \cup [\overline{T}_2 \setminus (T' \cup T_{g,2})]$ , where  $\overline{\phantom{x}}$  denotes closure.
- Let  $T'_g$  be a tubular neighbourhood around the strict transform of  $g$ , chosen small with respect to the above.

**Definition 3.2.** We define  $F_{f,g}$  to be a twisting of  $T_{f,g}$  along the strict transform of  $g$ . More precisely, for any  $a \in \mathcal{A}_g$ , the set  $E_a \cap T_{f,g}$  is a union of  $m_w$  disks embedded in  $T_{f,g}$  as in 2.4, where  $w \in \mathcal{W}$  so that  $(a, w) \in \mathcal{E}$ . Take the  $l_a$ -th twist along each of these disks.

**Theorem 3.3.** *The Milnor fibre  $F_\Phi$  is diffeomorphic to the space  $F_{f,g}$  constructed above. The monodromy can be chosen to satisfy the following*

- *The set  $\overline{T}_{f,1} \setminus T'$  is invariant under  $m_\Phi$  and the restriction is homotopic to the identity.*
- *We have  $m_\Phi|_{F_f} = m_f$*
- *The set  $T_2 \setminus (T' \cup \overline{T}_{g,2})$  is invariant under  $m_\Phi$  and the restriction is homotopic to the identity.*
- *For any  $a \in \mathcal{A}_{g,2}$ , the monodromy  $m_\Phi$  permutes the  $m_{w_a}$  handles corresponding to  $a$  cyclically.*

**Corollary 3.4.** (i) *The Euler characteristic of  $F$  is given by the formula*

$$\chi(F_\Phi) = \sum_{w \in \mathcal{W}_1} m_w(2 - \delta_{w,f}) + \sum_{a \in \mathcal{A}_{g,2}} m_{w_a}.$$

(ii) *The zeta function associated to  $\Phi$  is given by the formula*

$$\zeta_\Phi(t) = \left( \prod_{w \in \mathcal{W}_1} (1 - t^{m_w})^{\delta_{w,f}-2} \right) \left( \prod_{a \in \mathcal{A}_{g,2}} (1 - t^{m_{w_a}})^{-1} \right) \quad (2)$$

where  $\delta_{w,f}$  is the number of vertices in  $\mathcal{W} \cup \mathcal{A}_f$  connected to  $w$  by an edge.

*Proof.* By 2.2, it is enough to prove 2.

Consider first the action of  $m_\Phi$  on  $\overline{T}_{f,g} \setminus T_2$ . Using a similar argument as in 2.3, we see that the zeta function of the restriction is the same as that of the restriction to  $F_f \cap \overline{T}_1 \setminus T_2$ . An A'Campo type argument shows that this zeta function is

$$\prod_{w \in \mathcal{V}(\Gamma_1)} (1 - t^{m_w})^{\delta_{w,f} - 2}. \quad (3)$$

Consider now the set  $\overline{T}_2 \cap \overline{T}_{f,g}$ . It has the homotopy type of a 3-manifold with some solid tori removed. In particular,  $\chi(\overline{T}_2 \setminus T') = 0$ . We can now use the same proof as that of ii to see that the zeta function of the restriction to  $T_{f,g} \cap \overline{T}_2$  is  $(\prod_{a \in \mathcal{A}_{g,2}} (1 - t^{m_{w_a}})^{-1})$ .

The intersection  $(\overline{T}_{f,g} \setminus T_2) \cap (\overline{T}_{f,g} \cap \overline{T}_2)$  is a disjoint union of circles which are cyclically permuted by the monodromy. The zeta function of the monodromy restricted to these circles is therefore 1.

Finally, using 2.1, we can glue these zeta functions together to get 2.  $\square$

**Corollary 3.5.** (i) If  $m_w \leq l_w$  for all  $w \in \mathcal{W}(\Gamma)$ , then  $F$  and  $F_{f,sing}$  have the same homotopy type and  $\zeta_\Phi = \zeta_f$ .

(ii) If  $m_w > l_w$  for all  $w \in \mathcal{W}(\Gamma)$ , then  $F$  has the same homotopy type as  $\vee_{m-1} S^2$ , where  $m = \sum_{a \in \mathcal{A}_g(\Gamma_2)} m_{w_a}$ . The zeta function is given by  $\zeta_\Phi(t) = \prod_{a \in \mathcal{A}_g} (1 - t^{m_{w_a}})$ .

*Proof.* In the case of i we have  $T_2 = \emptyset$ . Twisting the handles corresponding to elements  $a \in \mathcal{A}_g$  does not alter the homotopy type. Therefore,  $F_\Phi$  has, by 3.3, the same homotopy type as  $\overline{T}_\epsilon \cup \overline{T}_{f,1}$ . Homotopically, this space is the same as  $F_f$ , where we have glued the boundary components to some circles. This can easily be seen as the same construction of  $F_{f,sing}$ . Furthermore, this homotopy equivalence can be seen as invariant under the actions of  $m_\Phi$  and  $m_f$ , proving  $\zeta_\Phi = \zeta_{f,sing}$  and thus  $\zeta_\Phi = \zeta_f$  by 2.3. For a second proof of this statement, one may compare A'Campo's formula for  $\zeta_f$  with 3.4.

In the case of ii,  $\mathcal{A}_{f,1} = \emptyset$ . We have  $A = \overline{T}_2 \setminus (T' \cup T_{g,2})$  and  $\overline{T}_2 = \overline{T}$ . Also,  $\overline{T} \setminus \overline{T}'$  is homotopically just  $\partial \overline{T} = S^3$  because the graph  $\Gamma$  describes a modification of the smooth germ  $(\mathbb{C}^2, 0)$ . In fact,  $\overline{T} \setminus T'$  is a collar neighbourhood around  $\partial \overline{T}$ , so  $\overline{T} \setminus T' = S^3 \times I$ . Furthermore, for  $a \in \mathcal{A}_g$ , the pair  $(\overline{T} \setminus T', \overline{T}_a \cap (\overline{T} \setminus T'))$  is isomorphic to the pair  $(S^3 \times I, S \times I)$  where  $S \subset S^3$  is a solid torus. Therefore,  $A$  is homotopically  $S^3$  with some solid tori removed, one for each element of  $\mathcal{A}_g$ . What's more, the attaching spheres of the handles are meridians of these tori. But removing a solid torus from a 3 manifold and adding  $m$  handles attached to meridians is equivalent to removing  $m$  spheres from the original manifold. This gives the same as  $\vee_{m-1} S^2$ .

The statment about  $\zeta_\Phi$  follows from 3.5  $\square$

**Example 3.6.** Let  $f(x, y) = x^d$  and  $g = y^d$  where  $d \geq 2$ . Then we can choose the resolution  $V$  so that  $\mathcal{V}$  has a single element, say  $\mathcal{V} = \{v\}$ . Then  $m_v = l_v = d$ , so we can apply 3.5a. The Milnor fiber  $F$  associated to  $\Phi$  has the same homotopy type as  $F_{f,sing}$ , which is up to homotopy a bouquet of  $d - 1$  two-spheres. Note that in spite of this,  $\Phi$  is not isolated. The zeta function of this singularity is  $\zeta(t) = t^d - 1$ .

## 4 Proof of theorem 3.3

To prove theorem 3.3 we project the embedded resolution  $V \times \mathbb{C} \rightarrow \mathbb{C}^3$  down to  $V$ , and study the image of the fiber  $F'_\Phi$ . Denote the projection by  $p$ . Choose  $r \in V$  with the property that  $g'(r) \neq 0$ . Assume further that there exists a  $z \in \mathbb{C}$  such that  $\Phi'(r, z) = \epsilon$ . We can solve this equation for  $z$ , namely

$$z = \frac{\epsilon - f'(r)}{g'(r)}.$$

This means that  $p$  restricts to an injection  $F'_\Phi \setminus (\text{St}_g \times \mathbb{C}) \rightarrow V$ , where  $\text{St}_g$  is the strict transform of  $g$ . Define a function  $Z : V \setminus \text{St}_g \rightarrow \mathbb{C}$  by  $Z(r) = (\epsilon - f'(r))/g'(r)$ . Instead of using the standard ball  $B_6 \subset \mathbb{C}^3$ , we can assume that  $F'_\Phi$  is simply defined as the set of pairs  $(r, z) \in V \times \mathbb{C}$  where  $r \in T$  and  $|z| \leq \delta$ . This way, we get a diffeomorphism  $F'_\Phi \setminus \text{St}_g \times \mathbb{C} \rightarrow X$  where

$$X = \{r \in V \setminus \text{St}_g : |Z(r)| \leq \delta\}.$$

We obtain a description of  $F_\Phi = F'_\Phi$  by considering the sets  $F'_\Phi \cap p^{-1}(\overline{T} \setminus T_g)$  and  $F'_\Phi \cap p^{-1}(\overline{T}_g)$ , and how they glue together along their intersection.

**Theorem 4.1.** *The following items determine the Milnor fibre and the monodromy.*

- (i) *Let  $e = (a, w) \in \mathcal{E}$  where  $a \in \mathcal{A}_{f,1}$  and  $w \in \mathcal{W}$ . There is a diffeomorphism between  $p(F'_\Phi) \cap \overline{T}_e$  and  $\overline{T}_e \setminus T'$  inducing identity on  $F'_f \cap \partial \overline{T}_e$  and its normal bundle in  $\partial \overline{T}_e$ .  
The set  $F'_\Phi \cap p^{-1}(\overline{T}_e)$  is invariant under the monodromy, up to homotopy the monodromy action is trivial on this set.*
- (ii) *The set  $p(F'_\Phi) \cap [\overline{T}_1 \setminus (T_f \cup T'_g \cup T_2)]$  is a tubular neighborhood around  $F'_f \cap [\overline{T}_1 \setminus (T_f \cup T'_g \cup T_2)]$  in  $T_1 \setminus (T_f \cup T'_g \cup T_2)$ .  
The set  $F'_\Phi \cap p^{-1}[\overline{T}_1 \setminus (T_f \cup T'_g \cup T_2)]$  is invariant under the monodromy. It can be chosen to coincide with  $m_f$  on the subset  $F'_f \cap [\overline{T}_1 \setminus (T_f \cup T'_g \cup T_2)]$  which is a strong homotopy retract.*
- (iii) *There is a diffeomorphism between  $p(F'_\Phi) \cap \overline{T}_2 \setminus T'_g$  and  $\overline{T}_2 \setminus (T' \cup T'_g)$  inducing identity on  $F'_f \cap \partial(T' \cup T'_g)$  and its normal bundle in  $\partial(T' \cup T'_g)$ . This set is invariant under the monodromy; its action is trivial up to homotopy.*
- (iv) *Let  $e = (a, w) \in \mathcal{E}$  where  $a \in \mathcal{A}_g$  and  $w \in \mathcal{W}$ . The set  $p^{-1}(\overline{T}'_g) \cap F'_\Phi$  is a disjoint union of  $m_w$  4 dimensional 2-handles glued to the manifold  $p(F'_\Phi) \setminus T'_g$ . The attaching spheres are those boundary components of  $F'_f \cap (\overline{T} \setminus T'_e)$  which are in  $\overline{T}'_e$ . The normal bundle of the attaching spheres has a canonical trivialisation since each component is the boundary of a disk in  $T'_e$ . The handles are attached with the  $(-l_a)$ -th framing. These handles are invariant under the monodromy, its action permutes them cyclically.*

*Proof of i.* We can choose coordinates around the point  $E_w \cap E_a$  so that  $f'(u, v) = u^m v^n$  and  $g(u, v) = u^l$ , where  $m = m_w$ ,  $n = m_a$  and  $l = l_w$ .



We can also suppose that  $\bar{T}_e = \{(u, v) : |u|, |v| \leq \rho\}$  where  $\rho$  is some number so that  $\epsilon \ll \rho$ . By choices made, we have  $m \leq l$  and  $n > 0$ .

Consider the space  $\tilde{T}_e = \{(u, \tilde{v}) : |u|, |\tilde{v}|^{1/n} < \rho\}$  and the map  $\pi_e : \bar{T}_e \rightarrow \tilde{T}_e$  given by  $(u, v) \mapsto (u, \tilde{v}) = (u, v^n)$ . We have then maps

$$Z_e(u, v) = \frac{u^m v^n - \epsilon}{u^l}, \quad \tilde{Z}_e(u, \tilde{v}) = \frac{u^m \tilde{v} - \epsilon}{u^l}$$

satisfying  $\tilde{Z}_e \circ \pi_e = Z_e$ . The function  $|\tilde{Z}_e|^2$  has the divisor  $u^m \tilde{v} = \epsilon$  as a nondegenerate critical manifold of index 0. This holds on  $\tilde{T}_e$  as well as  $\partial \tilde{T}_e$ . We will show that  $|\tilde{Z}_e|^2$  has no other critical manifolds (in the interior or the boundary) in the preimage  $|\tilde{Z}_e|^2 \leq \delta^2$ . This will show that the set  $\tilde{F}_e = \pi_e(p(F_\Phi) \cap T_e)$  is a tubular neighbourhood around the submanifold given by  $u^m \tilde{v} = \epsilon$ , that is,  $\pi_e(F_f)$ . Note first that the coordinate  $u$  takes nonzero values on  $\tilde{F}_e$ , since  $Z_e$  has a pole along the exceptional divisor. We have then  $\partial_{\tilde{v}} Z_e(u, \tilde{v}) = u^{m-l} \neq 0$  on  $\tilde{F}_e$ . This shows that  $|\tilde{Z}_e|^2$  has no critical points in the interior  $\tilde{T}_e$ , nor on the part of the boundary given by  $|u| = \rho$ . For the rest of the boundary, we will show that if  $|\tilde{v}| = \rho^n$ , then  $\partial_u Z_e \neq 0$ . But we have

$$\partial_u \tilde{Z}_e = (m-l)u^{m-l-1}\tilde{v} + lu^{l-1}\epsilon = ((m-l)u^m \tilde{v} + \epsilon l)u^{l-1}.$$

If  $m = l$ , then this shows that the partial derivative does not vanish. Assuming  $m < l$  we find that  $\tilde{Z}_e(u, \tilde{v}) = 0$  implies  $u^m = -\epsilon l / ((m-l)\tilde{v})$ . This implies

$$|\tilde{Z}_e(u, \tilde{v})| = \left| \frac{-\frac{\epsilon l}{(m-l)\tilde{v}} - \epsilon}{-\frac{\epsilon l}{(m-l)\tilde{v}}} \right|^{1/m} = \left| \frac{l}{(m-l)\rho^n} - 1 \right| \left| \frac{(m-l)\rho}{l} \right|^{-l/m} \epsilon^{1-l/m},$$

so that  $|Z_e(u, \tilde{v})|$  is huge, since  $\epsilon$  is small and  $l > m$ . In particular,  $|\tilde{Z}_e| > \delta$ .

We have now showed that  $\tilde{F}_e$  is a tubular neighbourhood around the divisor  $u^m \tilde{v} = \epsilon$ . But the same is true about the set  $\tilde{T}_e \setminus \pi_e(T')$ . Thus, we have a diffeomorphism  $\tilde{\psi}_e : \tilde{F}_e \rightarrow \tilde{T}_e \setminus \pi_e(T')$  and we can assume that  $\tilde{\psi}_e$  equals the identity on a small neighbourhood around the divisor  $u_m \tilde{v} = \epsilon$ . Now, the set  $\{\tilde{v} = 0\} \cap \tilde{F}_e$  is an annulus given by  $|u| \geq |\epsilon/\delta|^{1/l}$ . One can now easily see that the map  $\tilde{\psi}_e$  can also be chosen to map this annulus into  $\pi_e(E_a) = \{\tilde{v} = 0\}$ . Finally, by considering the symmetries of  $\tilde{F}_e$  and  $\tilde{T}_e$ , one can assume that  $\tilde{\psi}_e$  commutes with multiplying  $\tilde{v}$  by a primitive  $n$ th root of unity. This, combined with the fact that  $F_e = \pi_e^{-1}(\tilde{F}_e)$ , shows that  $\tilde{\psi}_e$  transfers to a diffeomorphism  $\psi_e : F_e \rightarrow T_e$ .  $\square$

**Lemma 4.1.1.** *The map  $F'_\Phi \cap [(\bar{T}_1 \setminus (T_f \cup T_2)) \times \mathbb{C}] \rightarrow \bar{D}_\delta$ ,  $(r, z) \mapsto z$  is proper, with surjective derivative everywhere. The same holds for its restriction to the boundary.*

*Proof.* The map is proper, since its domain is compact. The surjectivity of the derivative requires more attention:

Let  $(r_0, z_0) \in F'_\Phi \cap [(\bar{T}_1 \setminus (T_f \cup T_2)) \times \mathbb{C}]$ . Then, we have three cases: in the first, there is a unique  $w \in \mathcal{W}_1$  so that  $r_0 \in T_w$ . Secondly, there might be exactly two elements  $w, w' \in \mathcal{W}_1$  such that  $r_0 \in T_w \cap T_{w'}$ . Thirdly, we might have  $r_0 \in T_e$  for some  $e = (w, a) \in \mathcal{E}$  for some  $w \in \mathcal{W}_1$  and  $a \in \mathcal{A}_{g,1}$ . In any case, we can find a coordinates  $u, v$  in a neighbourhood  $U$  around  $r_0$  in

$V$  such that  $f'(u, v) = u^m v^l$  and  $g'(u, v) = \alpha u^l v^k$  for  $m = m_w$ ,  $l = l_w$  and some non-vanishing function  $\alpha : U \rightarrow \mathbb{C}$ . We have  $m \leq l$ , and one out of three, depending on the cases above:  $n = k = 0$ ,  $n \leq k$  or  $n = 0, k > 0$ . In any case, we have  $n \leq k$ .

By the inverse function theorem, the map  $F'_\Phi \cap U \rightarrow \mathbb{C}^2$ ,  $(u, v, z) \rightarrow (v, z)$  is a coordinate chart, provided that  $\partial_u \Phi' \neq 0$  on  $F'_\Phi \cap U$ . We have

$$\begin{aligned} \partial_u \Phi'(u, v, z) &= \partial_u(u^m v^n + z \alpha u^l v^k) = m u^{m-1} v^n + z((\partial_u \alpha) u^l v^k + \alpha l u^{l-1} v^k) \\ &= u^{m-1} v^n (m + z u^{l-m} v^{k-n} (\partial_u \alpha u + \alpha l)). \end{aligned}$$

The function  $u^{l-m} v^{k-n} (\partial_u \alpha u + \alpha l)$  is continuous, and therefore bounded on  $U$  (we can assume that  $U$  is relatively compact). Since  $|z| \leq \delta$ , we get

$$|z u^{l-m} v^{k-n} (\partial_u \alpha u + \alpha l)| \ll m.$$

proving that  $\partial_u \Phi' \neq 0$  on  $F'_\Phi \cap (U \times \mathbb{C})$ . Therefore, the function  $z$  is a part of a coordinate system around  $(r_0, z_0)$ . In particular, its derivative is surjective.

For the last statement, the same reasoning applies; the equation  $\partial_u \Phi' \neq 0$  implies that  $z$  (as two real variables) gives part of a coordinate system on the boundary. We omit the details.  $\square$

*Proof of ii.* The argument in the proof of 4.1.1 can be transferred directly to the boundary components  $p(F'_\Phi) \cap \partial \overline{T}_{g,1}$ . We can therefore use Ehresmann's fibration theorem to get that the restriction of  $Z$  to the set  $p(F'_\Phi) \cap [\overline{T}_1 \setminus (T_f \cup T_g \cup T_2)]$  is a locally trivial fibration over  $D_\delta$ . Since  $D_\delta$  is contractible, this fibration is trivial. The fiber over  $0 \in D_\delta$  is simply  $F'_f \cap [\overline{T}_1 \setminus (T_f \cup T_g \cup T_2)]$ . Therefore, the set  $p(F'_\Phi) \cap [\overline{T}_1 \setminus (T_f \cup T_g \cup T_2)]$  is a product  $F'_f \cap [\overline{T}_1 \setminus (T_f \cup T_g \cup T_2)] \times D$ . This proves the statement.  $\square$

**Lemma 4.1.2.** *We may assume that the inequality  $|g/f| < \delta/2$  holds in  $\overline{T}_2$ .*

*Proof.* Let  $x, y$  be some generically chosen coordinates on  $\mathbb{C}^2$ . In  $\overline{T}_1$  (in all of  $V$ , for that matter) we have  $|x'|, |y'| \leq \delta$ , where  $x', y'$  are the pullbacks of  $x, y$ . In  $T_2$ , the function  $f'/g'$  vanishes along  $E \cap \overline{T}_2$  by definition of  $\mathcal{W}_2$ . Since  $x'$  and  $y'$  vanish with order one along  $E$  we have  $|f'/g'| \leq C \|(x', y')\|$  in  $\overline{T}_2$  for some  $C > 0$ . Multiplying  $f$  with  $C^{-1}$ , however, gives an equivalent singularity because the germ  $C^{-1}f + zg = C^{-1}(f + Czg)$  is equivalent with the germ  $f + zg$  via the coordinate change  $(x, y, z) \leftrightarrow (x, y, Cz)$ .  $\square$

*Proof of iii.* We start by investigating the intersection of  $p(F'_\Phi)$  with the smaller set  $\overline{T}_2 \setminus (T_1 \cup T_g)$ . The remaining parts will be considered separately.

As before, we have

$$p(F'_\Phi) \cap \overline{T}_2 \setminus (T_1 \cup T_g) = \{r \in \overline{T}_2 \setminus (T_1 \cup T_g) : |Z(r)| \leq \delta\}.$$

We will start by showing that  $|Z|^{-1}$  is a Morse-Bott function in  $\overline{T}_2 \setminus (T_1 \cup T_g)$  which defines a small tubular neighbourhood around the exceptional divisor. More precisely, let

$$N = \{r \in \overline{T}_2 \setminus (T_1 \cup T_g) : |Z(r)|^{-1} < \delta^{-1}\}.$$

We will prove that  $N$  is a tubular neighbourhood around the exceptional divisor in  $\overline{T}_2 \setminus (T_1 \cup T_g)$ , and that it can be made arbitrarily small by shrinking  $\epsilon$ . The restriction of  $g'$  to  $\overline{T}_2 \setminus (T_1 \cup T_g)$  is a holomorphic function vanishing exactly on the exceptional divisor. Therefore, the set  $\{r \in \overline{T}_2 \setminus (T_1 \cup T_g) : |g'(r)| \geq 2\epsilon/\delta\}$  is the complement of a small neighbourhood around the exceptional divisor. If  $r \in \overline{T}_2 \setminus (T_1 \cup T_g)$  satisfies  $|g'(r)| \geq 2\epsilon/\delta$ , we get

$$|Z(r)| = \left| \frac{f'(r) - \epsilon}{g'(r)} \right| \leq \left| \frac{\epsilon}{g'(r)} \right| + \left| \frac{f'(r)}{g'(r)} \right|.$$

By the choice of  $r$ , we have  $\epsilon/|g'(r)| < \delta/2$ . By 4.1.2, we also have  $|f'(r)/g'(r)| \leq \delta/2$ . Therefore, we get  $|Z(r)| \leq \delta$ . We have proven

$$N \subset N' := \{r \in V : |g'(r)| \leq 2\epsilon/\delta\} \setminus (T_1 \cup T_g).$$

The set  $N'$  above can be made arbitrarily small, as a neighbourhood around the exceptional divisor. To show that  $N$  is a tubular neighbourhood, we will prove that the derivative of  $Z$  does not vanish in  $N'$  outside the exceptional divisor. Choose coordinates  $u, v$  around  $r \in V$  such that  $f'(u, v) = u^m v^l$  and  $g'(u, v) = \alpha u^l v^k$  where  $m = m_w$ ,  $l = l_w$  for some  $w \in \mathcal{W}_2$  for which  $r \in \overline{T}_w$  and either there is a  $w' \in \mathcal{W}_2$  so that  $n = m_{w'}$  and  $k = l_{w'}$ , or  $n = l = 0$ . In any case, we have  $m > l$  and  $n \geq k$ . We calculate:

$$\partial_u Z(r) = \frac{\partial}{\partial u} \frac{u^m v^n - \epsilon}{\alpha u^l v^k} = \frac{m u^{m-1} v^n \alpha u^l v^k - (u^m v^n - \epsilon)((\partial_u \alpha) u^l - \alpha l u^{l-1}) v^k}{(\alpha u^l v^k)^2}.$$

Simplifying, we get  $\partial_u Z(r) \neq 0$  if and only if

$$m u^m v^n \alpha - (u^m v^n - \epsilon)((\partial_u \alpha) u - \alpha l) \neq 0. \quad (4)$$

By assumption, we have  $|Z(r)| > \delta$ , that is,  $|u^m v^n - \epsilon| > \delta |\alpha u^l v^k|$ . Thus, we prove 4 by showing that

$$|m u^{m-l} v^{n-k}| < \delta |(\partial_u \alpha) u - \alpha l|. \quad (5)$$

The number  $|(\partial_u \alpha) u - \alpha l|$  is bounded below independent of  $\delta$  and  $\epsilon$ , because  $|u|$  is small with respect to  $\alpha l$ , which is bounded below. The functions  $|u^{m-l} v^{n-k}|$  and  $g'(u, v)$  have the same zero set, thus there is a  $C, \gamma \in \mathbb{R}_+$  so that  $|m u^{m-l} v^{n-k}| < C |g'(u, v)|^\gamma \leq C (2\epsilon/\delta)^\gamma \ll \delta$ . This proves 5. Hence, the set  $N$  is a tubular neighborhood around the exceptional divisor in  $\overline{T}_2 \setminus (T_2 \cup T_g)$ .

Now consider an edge  $e = (w_1, w_2) \in \mathcal{E}$  where  $w_i \in \mathcal{W}_i$ . We want to prove that there is a diffeomorphism between  $T_e \cap p(F'_\Phi)$  and  $\overline{T}_e \setminus T'$  fixing the intersection  $F'_f \cap \partial \overline{T}_e$  and its normal bundle inside  $\partial \overline{T}_e$ .

Consider coordinates  $u, v$  on  $T_e$  so that  $f = u^m v^n$  and  $g = u^l v^k$ . Let  $\tau_1, \tau_2 \in \mathbb{C}$  with  $|\tau_1| = 1$ . Then the set  $\{(u, v) \in \overline{T}_e : \text{Arg}(u) = \tau_1, \text{Arg}(v) = \tau_2, |Z(u, v)| \leq \delta\}$  is a disk. In fact:

- If  $\tau_1^m \tau_2^n \neq 1$ , then  $|Z|^2$  restricts to a Morse function on the manifold  $\{(u, v) : \text{Arg}(u) = \tau_1, \text{Arg}(v) = \tau_2\}$ . There are no critical points on the interior. Restricting  $Z$  to the boundary of this submanifold, we get exactly one critical point with index zero and at most one with index one.

- If  $\tau_1^m \tau_2^n = 1$ , then  $|Z|^2$  restricts to a Morse-Bott function on the submanifold  $\{(u, v) : \text{Arg}(u) = \tau_1, \text{Arg}(v) = \tau_2\}$ , the critical set being the intersection with  $F'_f$ .

Proving these two statements is a simple exercise, it boils down to showing that certain partial derivatives do not vanish. The results show that each fiber of the argument map  $(\tau_1, \tau_2)$  is abstractly a disk. One is therefore free to choose a diffeomorphism from this disk to the set of points  $(u, v)$  where the argument of each coordinate is fixed, to the set of points with corresponding arguments in  $\overline{T}_e \subset T'$ . This can be done in such a way that we get a diffeomorphism with the desired properties.

The last thing we need to consider is the set  $p(F'_\Phi) \cap (\overline{T}_{g,2} \setminus T'_{g,2})$ . Let  $a$  be in  $\mathcal{A}_{g,2}$ . We have local coordinates  $u, v$  on  $T_a$  so that  $f = u^m$  and  $g = u^l v^k$ , where  $m = m_{w_a}$ ,  $l = l_{m_w}$  and  $k = m_a$ . The set  $p(F'_\Phi) \cap (\overline{T}_a \setminus T'_a)$  can be given by equations  $|Z| \leq \delta$  and  $|v| \geq \eta$  for some  $\eta \ll \epsilon$ , that is,

$$\left\{ (u, v) \in \overline{T}_a : \left| \frac{u^m - \epsilon}{u^l v^k} \right| \leq \delta, |v| \geq \eta \right\}$$

We proved already, that the intersection  $p(F'_\Phi) \cap \{|v| = \rho_a\}$  is the complement of a tubular neighbourhood around the exceptional divisor in the set  $\{|v| = \rho_a\}$ . Take a point  $(u_0, v_0) \in p(F'_\Phi) \cap \overline{T}_a \setminus T'_a$ . From the formula  $Z(u, v) = (u^m - \epsilon)/(u^l v^k)$  we see that the segment between  $(u_0, v_0)$  and  $(u_0, (\rho_a/|v_0|)v_0)$  is contained in  $p(F'_\Phi) \cap (\overline{T}_a \setminus T'_a)$ . From this, one quickly observes that the inclusion  $p(F'_\Phi) \cap (\overline{T}_a \setminus T'_a) \rightarrow \overline{T}_a \setminus T'_a$  is isotopic to the inclusion of  $(\overline{T}_a \setminus T'_a) \setminus T'$  fixing a neighborhood around both  $F'_f \cap (\overline{T}_a \setminus T'_a)$  and  $\{|v| = \rho_a\}$ .

Finally, all these diffeomorphisms glue together to the desired map.

For the monodromy, we notice that the diffeomorphism type of the pair  $(F'_{\Phi, \theta}, p^{-1}(\overline{T}_2 \setminus (T_1 \cup T_g)) \cap F'_{\Phi, \theta})$ , where  $\theta \in S^1$  and  $F'_{\Phi, \theta} = \Phi'^{-1}(\theta\epsilon)$ , is independent of  $\theta$ , that is,  $p^{-1}(\overline{T}_2 \setminus (T_1 \cup T_g)) \cap F'_{\Phi, \theta}$  is a subbundle of the Milnor fibration. The description of this fibre above is independent of  $\theta$  however, and therefore gives a trivialisation of the bundle. Therefore, the monodromy acts trivially, up to homotopy, on this subset.  $\square$

*Proof of iv.* Let  $a \in \mathcal{A}_g$ . As before, we consider coordinates  $u, v$  on  $\overline{T}_a$  so that  $f' = u^m$  and  $g' = u^l v^k$ . Then  $H_a := F'_\Phi \cap p^{-1}(\overline{T}'_a)$  is the set of points  $(u, v, z)$  satisfying  $|z| \leq \delta$ ,  $|v| \leq \eta$  for some  $\eta \ll \epsilon$  and the equality  $\Phi' = \epsilon$ . We show first that abstractly, this set is a disjoint union of bidisks. Clearly, the map  $\pi_a = (v, z) : H_a \rightarrow D_\eta \times D_\delta$  is a proper surjection which maps boundary points to boundary points. Also, the preimage of  $(0, 0)$  is the set  $\{(u, 0, 0) : u^m = \epsilon\}$ , and so contains exactly  $m$  points. By the implicit function theorem, if  $\partial_u \Phi' \neq 0$  on  $H$ , the map  $\pi_a$  is a local diffeomorphism, and so a covering map. Furthermore, since the bidisk is contractible, such a covering map must be a product. We get

$$\partial \Phi'(u, v, z) = \partial_u(u^m + zu^l v^k) = mu^{m-1} + zu^{l-1} v^k.$$

The functions  $|mu^{m-1}|$  is bounded below by a positive number on  $H_a$ , since it is continuous and does not vanish. Similarly, the function  $|zu^{l-1} v^k|$  is bounded

above. Taking  $\eta$  small enough, we get  $|mu^{m-1}| > |zlu^{l-1}v^k|$  on  $H_a$ . This gives  $\partial_u \Phi' \neq 0$  as required.

We have now shown that  $F'_\Phi$  is given by glueing handles (such as  $h$ ) to  $T_{f,g} \setminus T'_g$  in the way described in 2.4. We only have to determine the twisting coefficient. We already have a parametrization of the handle  $h$  by  $(u, v)$ . The handle already contained in  $T_{f,g}$  is parametrized by  $(u - \xi, v)$ , where  $\xi$  is some  $m$ -th root of unity. Denote this parametrization by  $\psi : \bar{D} \times \bar{D} \rightarrow T_{f,g}$ .

Now, for any  $r \in h$  with coordinates  $(z, v)$  we have  $p(r) = (U(z, v), v)$  where

$$zU(z, v)^l v^k = U(z, v)^m - \epsilon$$

and we assume that  $U(z, v)$  is in a small neighbourhood around some  $m$ -th root of  $\epsilon$ . This shows that the twisting coefficient used to glue  $h$  is  $k$ , as stated.

To finish the proof, we must consider the action of the monodromy on the handles corresponding to  $a \in \mathcal{A}_{g,2}$ . But the central disks of these handles are given by  $F'_f$ . It follows that they are permuted cyclically.  $\square$

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